

ON A THEOREM OF KISIN

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Let K be a p -adic field, i.e., a complete discretely valued field of characteristic 0 with perfect residue field of characteristic $p > 0$, and \bar{K} be an algebraic closure of K . We fix a uniformiser $\pi \in K$. Let $\Xi = \Xi_\pi$ be the corresponding Kummer $\mathbb{Z}_p(1)$ -torsor; its elements are sequences $\xi = (\xi_n)_{n \geq 0}$ of elements in \bar{K} such that $\xi_{n+1}^p = \xi_n$, $\xi_0 = \pi$. Pick one ξ , and set $K_\xi = \cup K(\xi_n)$. Consider the Galois groups $G := \text{Gal}(\bar{K}/K)$, $G_\xi := \text{Gal}(\bar{K}/K_\xi)$; let $\text{Rep}(G)$, $\text{Rep}(G_\xi)$ be the categories of their finite-dimensional \mathbb{Q}_p -representations.

The next result was conjectured by Breuil [B] and proved by Kisin [K] 0.2; the proof in loc. cit. is based on theory of Kisin modules. This note provides an alternative argument that uses only basic properties of Fontaine's rings; its key ingredient (namely, (i) of the lemma below) is the same as in Grothendieck's proof of the monodromy theorem.

Theorem. *The restriction functor $\text{Rep}(G) \rightarrow \text{Rep}(G_\xi)$ is fully faithful on the subcategory of crystalline representations.*

Proof. The Galois group G acts on Ξ , and G_ξ is the stabilizer of ξ . The action is transitive, i.e., $G/G_\xi \xrightarrow{\sim} \Xi$, since polynomials $t^{p^n} - \pi$ are irreducible.

Let R be the ring of continuous \mathbb{Q}_p -valued functions on Ξ . Let $R_{\text{st}} \subset R_\phi$ be the subrings of polynomial, resp. locally polynomial, functions (this makes sense since Ξ is $\mathbb{Z}_p(1)$ -torsor). Since G acts on Ξ by affine transformations, its action on R preserves the subrings.

Lemma. (i) R_ϕ is the union of all finite-dimensional G -submodules of R .
(ii) R_{st} is the union of all semi-stable G -submodules of R_ϕ .
(iii) \mathbb{Q}_p is the only nontrivial crystalline G -submodule of R_{st} .

Assuming the lemma, let us prove the theorem. For $V \in \text{Rep}(G_\xi)$ we denote by $I(V)$ the induced G -module, that is the space of all continuous maps $f : G \rightarrow V$ such that $f(hg) = hf(g)$ for $h \in G_\xi$, the action of G is $g(f)(g') = f(g'g)$. It is a G -equivariant R -module, the R -action is $(rf)(g) = r(g^{-1}\xi)f(g)$. For $U \in \text{Rep}(G)$ we have the Frobenius reciprocity $\text{Hom}_{G_\xi}(U, V) \xrightarrow{\sim} \text{Hom}_G(U, I(V))$ that identifies $\alpha : U \rightarrow V$ with $\tilde{\alpha} : U \rightarrow I(V)$, $\tilde{\alpha}(u)(g) = \alpha(gu)$, $\alpha(u) = \tilde{\alpha}(u)(1)$. For $V \in \text{Rep}(G)$ the image of $\text{id}_V \in \text{Hom}_{G_\xi}(V, V)$ is a G -morphism $V \rightarrow I(V)$ that yields an identification of G -equivariant R -modules $V \otimes R \xrightarrow{\sim} I(V)$.

So for $V_1, V_2 \in \text{Rep}(G)$ one has identifications $\text{Hom}_{G_\xi}(V_1, V_2) = \text{Hom}_G(V_1, I(V_2)) = \text{Hom}_G(V_1, V_2 \otimes R) = \text{Hom}_G(V_1 \otimes V_2^*, R) = \text{Hom}_G(V_1 \otimes V_2^*, R_\phi)$, the last equality

The first named author was supported in part by NSF Grant DMS-0401164.

comes from (i). If both V_i are crystalline, then this equals $\mathrm{Hom}_G(V_1 \otimes V_2^*, \mathbb{Q}_p) = \mathrm{Hom}_G(V_1, V_2)$ by (ii), (iii). Thus $\mathrm{Hom}_{G_\xi}(V_1, V_2) = \mathrm{Hom}_G(V_1, V_2)$, q.e.d. \square

Proof of Lemma. Let P be the group of all affine automorphisms of $\mathbb{Z}_p(1)$ -torsor Ξ ; it is an extension of \mathbb{Z}_p^\times by $\mathbb{Z}_p(1)$, the choice of ξ gives a splitting. Let $\eta : G \rightarrow P$ be the action of G on Ξ ; its composition with $P \twoheadrightarrow \mathbb{Z}_p^\times$ is cyclotomic character χ .

Consider the filtration $R_{\mathrm{st} \, n}$ on R_{st} by the degree of the polynomial. Then G acts on $\mathrm{gr}_n R_{\mathrm{st}}$ by χ^{-n} , i.e., $\mathrm{gr}_n R_{\mathrm{st}}$ is isomorphic to $\mathbb{Q}_p(-n)$.

There is a canonical morphism $\varepsilon : R_{\mathrm{st}} \rightarrow B_{\mathrm{st}}$ of \mathbb{Q}_p -algebras defined as follows. For $\xi \in \Xi$ let $l_\xi : \Xi \rightarrow \mathbb{Z}_p(1)$ be the identification of torsors such that $l_\xi(\xi) = 0$. If τ is a generator of $\mathbb{Z}_p(1)$, then $\tau^{-1}l_\xi \in R_{\mathrm{st}}$ is a linear polynomial function, i.e., a free generator of R_{st} . We define ε by formula $\varepsilon(\tau^{-1}l_\xi) = -\tau^{-1}\lambda(\xi)$. Here in the r.h.s. we view τ as an invertible element of B_{crys} via the embedding $\mathbb{Z}_p(1) \hookrightarrow B_{\mathrm{crys}}$ from [F1] 2.3.4, and $\lambda(\xi) \in B_{\mathrm{st}}$ is as in [F1] 3.1.4. It follows from the definitions in [F1] 3.1 that ε does not depend on the auxiliary choice of ξ . It evidently commutes with the Galois action. Since $\log(\xi)$ is a free generator of B_{st} over B_{crys} , we see that ε is injective and $R_{\mathrm{st} \, n}$ for $n \geq 1$ are *non-crystalline* semi-stable G -modules.

Choose v and \log from [F2] 5.1.2 as $v(\pi) = 1$, $\log(\pi) = 0$. As in [F2] 5.2, this yields the fully faithful tensor functor $D_{\mathrm{st}} : \mathrm{Rep}(G)_{\mathrm{st}} \rightarrow \mathrm{MF}_K(\varphi, N)$.

Consider the polynomial algebra $K_0[t]$. We equip it with Frobenius semi-linear automorphism φ , $\varphi(t) := pt$, the K_0 -derivation $N := \partial_t$, and the Hodge filtration $F^i :=$ the K -span of $t^{\geq i}$. The subspaces of polynomials of degree $\leq n$ are filtered (φ, N) -modules, so $K_0[t]$ is a ring ind-object of $\mathrm{MF}_K(\varphi, N)$.

There is a canonical isomorphism $K_0[t] \xrightarrow{\sim} D_{\mathrm{st}}(R_{\mathrm{st}})$ which identifies t with $(\tau^{-1}l_\xi) \otimes \tau + 1 \otimes \lambda(\xi) \in (R_{\mathrm{st}} \otimes B_{\mathrm{st}})^G = D_{\mathrm{st}}(R_{\mathrm{st}})$. Thus each $D_{\mathrm{st}}(R_{\mathrm{st} \, n})$ is a single Jordan block for the action of N , so every finite-dimensional G -submodule of R_{st} equals one of $R_{\mathrm{st} \, n}$'s, which implies (iii).

Notice that $R_\phi = R_0 \otimes R_{\mathrm{st}}$, where R_0 is the subring of locally constant functions. Since G acts transitively on Ξ , one has $R_0^G = \mathbb{Q}_p$ and all finite-dimensional G -modules that occur in R_0 are generated by G_ξ -fixed vectors. These representations are Artinian, hence semisimple, so we have the decomposition $R_0 = \mathbb{Q}_p \oplus R'_0$. Since the map $G_\xi \rightarrow \mathrm{Gal}(K^{\mathrm{un}}/K)$, where $K^{\mathrm{un}} \subset \bar{K}$ is the maximal unramified extension of K , is surjective (for $K^{\mathrm{un}} \cap K_\xi = K$), every G -module in R'_0 is ramified. Thus every irreducible subquotient of $R'_0 \otimes R_{\mathrm{st}}$ is *not* semi-stable, and we get (ii).

It remains to prove (i). We first show that $\eta(G)$ is an open subgroup of P . Since $\chi(G)$ is open in \mathbb{Z}_p^\times , it suffices to check that $\eta(G) \cap \mathbb{Z}_p(1)$ is open in $\mathbb{Z}_p(1)$. Since every closed nontrivial subgroup of $\mathbb{Z}_p(1)$ is open, we need to check that $\eta(G) \cap \mathbb{Z}_p(1) \neq \{0\}$. If not, then $\eta(G) \xrightarrow{\sim} \chi(G)$ is commutative, so G acts on R through an abelian quotient. This implies, since $\mathrm{gr}_n R_{\mathrm{st}} \simeq \mathbb{Q}_p(-n)$ are pairwise non-isomorphic G -modules, that filtration $R_{\mathrm{st} \, n}$ splits, which is not true, q.e.d.

Let τ be a generator of $\mathbb{Z}_p(1) \subset P$; then R_ϕ is the union of all finite-dimensional $\mathbb{Z}_p(1)$ -submodules of R on which all eigenvalues of τ are roots of 1. Since $\eta(G)$ has finite index in P , it suffices to show that every finite-dimensional P -submodule V of R has this property. This follows since for $g \in P$ one has $g\tau g^{-1} = \tau^m$, where m is the image of g in \mathbb{Z}_p^\times , and there are only finitely many eigenvalues of τ on V . \square

The above proof was first found by F.T.R. and later and independently by A.B. We are grateful to Bhargav Bhatt, Matt Emerton, and Mark Kisin (A.B.), and to Laurent Berger and François Brunault (F.T.R.) for very helpful discussions.

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